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ANALYTICAL SOLITONS TO THE GENERALIZED KdV EQUATION BY ADOMIAN DECOMPOSITION METHOD

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Abstract

In this paper, enhanced Adomian decomposition method (ADM) is applied to study the generalized Korteweg-de Varies (gKdV) equation. We express, explicitly, the Adomian polynomials (APs) for the advection nonlinearity term in various orders, starting from the lowest-order quadratic term to the nonic order nonlinearity term. Then, the initial condition is expressed as a Taylor series and each term is distributed to terms in the integral equation that constitute the solitons solution in series form. Resulting to exact analytically-continuous solitons akin to Multivariate Taylor Theorem (MTT) applied to the exact solution, which is against the existing discrete ones currently available in literature. Our results from specific illustration were further depicted in 3D plots from Maple 2021 computer algebra system.

Keywords and phrases: Korteweg-de Varies equation, Taylor's theorem, Adomian polynomials, Adomian decomposition method.

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1. Introduction

The gKdV equation in the x-direction is a third order nonlinear hyperbolic partial differential equation with an introduced initial condition of the form

$$u_t + \alpha u^p u_x + \beta u_{xxx} = 0, \quad u(x, 0) = \phi(x) \tag{1}$$

which has broad ranging applications with a complete history in [15]. The equation is a universal mathematical model that describes weakly nonlinear long wave propagation in dispersive media and it arises in many field of science. A comprehensive list of the applications is contained in [10-13] and the literatures therein. The Solitons or Solitary waves, as described by [11], $u = u(x, t)$ is the wave profile that describes the elongation of the wave at a place in the space coordinate x in time, t, $p \in \mathbb{Z}^+$, α and β are determined by the wave medium properties which could either be constants or function of x, t. Nowadays, as can be seen in [1-16], α and β are considered as constants. From the earlier conceived KdV equation, now associated with variable change, coupled with some casting, has culminated into the canonical form

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2}$$

as contained in [11]. Which is also of quadratic nonlinearity and here 6 is a scaling factor making the solitons easier to describe.

Several approaches in literature have been developed while studying and presenting the KdV equation. Pioneer studies on analytical result were carried out by [14]. They developed the inverse scattering transform method to solve the KdV equation. [16] presented a numerical solution using B-Spline finite element method approach, [1] presented the polynomial solutions in terms of Jacobi's elliptic function and also utilises the extended expansion method. [2] presented a derivation of multiple scale expansion and multiple level approximation method, [25] studied the KdV equation using the exponential-expansion method and [3] studied the KdV equation with negative order hierarchy that generates nonlinear integrable equations. [4] studied the $(2+1)$-dimensional class of this equation by reducing the governing equation to simpler ordinary differential equations using wave transformation in association with Jacobi's elliptic function. [5] studied the negative order class of this equation in $(3+1)$-dimension, which was also followed suit by [6] and [22] studied the linearised version with reduced differential transform method using time dependent boundary conditions. [7] presented an extended study of Homotopy perturbation method to investigate the equation numerically while [8] presented the semi-implicit pseudo-spectral method and [9] presented the Homotopy analysis method to investigate the KdV equation numerically. [23] studied the KdV equation with quadratic nonlinearity and fifth-order in the space variable using the unified F-expansion method, and [24] presented an analysis of the approximate symmetries to the perturbed KdV equation with partial Lagrange method.

Studies on the KdV equation are still ongoing and more results are still evolving. So far, emphasis has mostly been focused on the quadratic nonlinearity term class of the equation with a hand full on the cubic and quartic nonlinearities. In this paper, we analyse the advection nonlinearity term with quadratic, cubic, quartic, ..., nonic nonlinearity terms using the APs in ADM. Then, decompose the given initial condition using the single variable Taylor's theorem, and apply the enhanced ADM on higher order nonlinearity cases $(p = 6, 7 \text{ and } 8)$ to provide continuous analytical wave profile as against the discrete cases currently available in literatures.

2. Theory of ADM on the gKdV Equation and its APs

2.1. ADM on the gKdV equation

The ADM in [17] as studied and presented in [19], [20] and [21] expresses the KdV equation (1) using the operator L_t and assumes L_t^{-1} exists. Then decomposes the wave profile $u(x, t)$ as $\sum_{n=0}^{\infty} u_n(x, t)$ and the advection nonlinearity term $\alpha u^p u_x$ as $Nu(x, t)$ with $Nu(x, t) = \sum_{n=0}^{\infty} A_n$, where, in this study, $L_t = \frac{\partial}{\partial t}(\cdot)$, $L_t^{-1} = \int_0^{\infty} (\cdot) dt$ and A_n represents the APs contained in [17, 19, 20, 21] and redefined in [18] as

$$A_n(u_0, u_0, u_0, \ldots) = \frac{1}{n} \frac{d^n}{d\lambda^n} \left[N \sum_{i=0}^{\infty} \lambda^i u_i(x, t) \right]_{\lambda=0} \tag{3}$$

$n \in (0 \cup \mathbb{Z}^+)$. On the whole, all the aforementioned dictates of the method optimally express equation (1) as

$$\sum_{n=0}^{\infty} u_n(x, t) = \phi(x) - \alpha L_t^{-1}(Nu(x, t)) - \beta L_t^{-1}(u_{xxx}(x, t)), \tag{4}$$

where, unlike the usual application of ADM, we write

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n(x)$$

which is a Taylor's series expansion of $\phi(x)$. Consequently, we obtain from equation (4) the following integral equations for each components of the Solitary wave.

$$u_0 = \phi_0(x),$$

$$u_1 = \phi_1(x) - L_t^{-1}\left[\beta \frac{\partial^3}{\partial x_3} u_0\right] - L_t^{-1}[\alpha A_n(u_0)],$$

$$u_2 = \phi_2(x) - L_t^{-1}\left[\beta \frac{\partial^3}{\partial x_3} u_1\right] - L_t^{-1}[\alpha A_n(u_0, u_1)],$$

$$u_3 = \phi_3(x) - L_t^{-1}\left[\beta \frac{\partial^3}{\partial x_3} u_2\right] - L_t^{-1}[\alpha A_n(u_0, u_1, u_2)],$$

...

$$u_k = \phi_k(x) - L_t^{-1}\left[\beta \frac{\partial^3}{\partial x_3} u_{k-1}\right] - L_t^{-1}[\alpha A_n(u_0, u_1, u_2, ..., u_{k-1})].$$

And, the wave profile is obtained as

$$u(x, t) = \lim_{k \to \infty} \sum_{n=0}^{k} u_n(x, t). \tag{5}$$

As a result of the infinite nature of equation (5), k is an integer of choice for the reseacher(s). The higher the value of k the more accurate the ADM solution. Some finite APs of quadratic, cubic, quartic, ..., nonic orders of $Nu(x, t)$ are given explicitly in Subsection 2.2.

2.2. Some APs of the advection nonlinearity term

In this section, we give some APs for the advection nonlinearity term.

(i) Quadratic nonlinearity term $(p = 1)$

$$A_0 = \alpha u_0 \frac{\partial}{\partial x} u_0,$$

$$A_1 = \alpha u_0 \frac{\partial}{\partial x} u_1 + \alpha u_1 \frac{\partial}{\partial x} u_0,$$

$$A_2 = \alpha u_0 \frac{\partial}{\partial x} u_2 + \alpha u_1 \frac{\partial}{\partial x} u_1 + \alpha u_2 \frac{\partial}{\partial x} u_0.$$

(ii) Cubic nonlinearity term $(p = 2)$

$$A_0 = \alpha u_0^2 \frac{\partial}{\partial x} u_0,$$

$$A_1 = \alpha u_0^2 \frac{\partial}{\partial x} u_1 + 2\alpha u_0 u_1 \frac{\partial}{\partial x} u_0,$$

$$A_2 = \alpha u_0^2 \frac{\partial}{\partial x} u_2 + 2\alpha u_0 u_1 \frac{\partial}{\partial x} u_1 + 2\alpha u_0 u_2 \frac{\partial}{\partial x} u_0 + \alpha u_1^2 \frac{\partial}{\partial x} u_0.$$

(iii) Quartic nonlinearity term $(p = 3)$

$$A_0 = \alpha u_0^3 \frac{\partial}{\partial x} u_0,$$

$$A_1 = \alpha u_0^3 \frac{\partial}{\partial x} u_1 + 3\alpha u_0^2 u_1 \frac{\partial}{\partial x} u_0,$$

$$A_2 = \alpha u_0^3 \frac{\partial}{\partial x} u_2 + 3\alpha u_0^2 u_1 \frac{\partial}{\partial x} u_1 + 3\alpha u_0^2 u_2 \frac{\partial}{\partial x} u_0 + 3\alpha u_0 u_1^2 \frac{\partial}{\partial x} u_0.$$

(iv) Quintic nonlinearity term $(p = 4)$

$$A_0 = \alpha u_0^4 \frac{\partial}{\partial x} u_0,$$

$$A_1 = \alpha u_0^4 \frac{\partial}{\partial x} u_1 + 4\alpha u_0^3 u_1 \frac{\partial}{\partial x} u_0,$$

$$A_2 = \alpha u_0^4 \frac{\partial}{\partial x} u_2 + 4\alpha u_0^3 u_1 \frac{\partial}{\partial x} u_1 + 4\alpha u_0^3 u_2 \frac{\partial}{\partial x} u_0 + 6\alpha u_0^2 u_1^2 \frac{\partial}{\partial x} u_0.$$

(v) Sixtic nonlinearity term $(p = 5)$

$$A_0 = \alpha u_0^5 \frac{\partial}{\partial x} u_0,$$

$$A_1 = \alpha u_0^5 \frac{\partial}{\partial x} u_1 + 5\alpha u_0^4 u_1 \frac{\partial}{\partial x} u_0,$$

$$A_2 = \alpha u_0^5 \frac{\partial}{\partial x} u_2 + 5\alpha u_0^4 u_1 \frac{\partial}{\partial x} u_1 + 5\alpha u_0^4 u_2 \frac{\partial}{\partial x} u_0 + 10\alpha u_0^3 u_1^2 \frac{\partial}{\partial x} u_0.$$

(vi) Septic nonlinearity term $(p = 6)$

$$A_0 = \alpha u_0^6 \frac{\partial}{\partial x} u_0,$$

$$A_1 = \alpha u_0^6 \frac{\partial}{\partial x} u_1 + 6\alpha u_0^5 u_1 \frac{\partial}{\partial x} u_0,$$

$$A_2 = \alpha u_0^6 \frac{\partial}{\partial x} u_2 + 6\alpha u_0^5 u_1 \frac{\partial}{\partial x} u_1 + 6\alpha u_0^5 u_2 \frac{\partial}{\partial x} u_0 + 15\alpha u_0^4 u_1^2 \frac{\partial}{\partial x} u_0.$$

(vii) Octic nonlinearity term $(p = 7)$

$$A_0 = \alpha u_0^7 \frac{\partial}{\partial x} u_0,$$

$$A_2 = \alpha u_0^7 \frac{\partial}{\partial x} u_1 + 7\alpha u_0^6 u_1 \frac{\partial}{\partial x} u_0,$$

$$A_2 = \alpha u_0^7 \frac{\partial}{\partial x} u_2 + 7\alpha u_0^6 u_1 \frac{\partial}{\partial x} u_1 + 7\alpha u_0^6 u_2 \frac{\partial}{\partial x} u_0 + 21\alpha u_0^5 u_1^2 \frac{\partial}{\partial x} u_0.$$

(viii) Nonic nonlinearity term $(p = 8)$

$$A_0 = \alpha u_0^8 \frac{\partial}{\partial x} u_0,$$

$$A_2 = \alpha u_0^8 \frac{\partial}{\partial x} u_1 + 8\alpha u_0^7 u_1 \frac{\partial}{\partial x} u_0,$$

$$A_2 = \alpha u_0^8 \frac{\partial}{\partial x} u_2 + 8\alpha u_0^7 u_1 \frac{\partial}{\partial x} u_1 + 8\alpha u_0^7 u_2 \frac{\partial}{\partial x} u_0 + 28\alpha u_0^6 u_1^2 \frac{\partial}{\partial x} u_0.$$

2.3. MTT for the wave profile

The wave profile $u(x, t)$ consists of the space variable x and the time variable t, by implication, has a multivariate Taylor's series representation. See [19, 21] and the literatures therein. First, we assume that $u(x, t)$ is a smooth function in a set, say $\mho$, with $(x_0, t_0) \in \mho$. Hence, we can write the wave profile as

$$u(x, t) = \sum_{m=0}^{\infty} \frac{1}{n!} \left[\left(\Delta x \frac{\partial}{\partial x} + \Delta t \frac{\partial}{\partial t} \right)^n u(x, t) \right]_{(x_0, t_0)}, \tag{6}$$

where $\Delta x = x - x_0$ and $\Delta t = t - t_0$.

Theorem 1 (Approximating Polynomials of the Solitons)**.** *Every* *smooth function $u(x, t)$ is a sum of approximating polynomials with given (x_0, t_0) and residual function.*

Proof. It follows from equation (6) that

$$u(x, t) = P_m(x, t) + R_{m+1}(x, t),$$

where $P_m(x, t)$ is the approximating mth-degree Taylor polynomials of $u(x, t)$ and $R_{m+1}(x, t)$ is the residual function which is much smaller for higher values of m. And, $R_{m+1}(x, t) \to 0$ as $(x, t) \to (x_0, t_0)$ making

$$u(x, t) \approx P_m(x, t). \qquad \square$$

3. Illustrations

In this section, we consider the gKdV equation as shown in equation (1) with $\alpha = \beta = 1$ and $\phi(x) = \left[A \operatorname{sech}^2(\kappa x - x_0)\right]^{\frac{1}{p}}$ that results to the exact solution $u(x, t) = \left[A \operatorname{sech}^2(\kappa x - ct - x_0)\right]^{\frac{1}{p}}$ as contained in [16], where $p \geq 1$, $A = \frac{2(p+1)(p+2)}{m^2}\kappa^2, \frac{4\kappa^2}{m^2}$. κ, m and x_0 are constants. We consider three cases

Case 1. Septic nonlinearity

Consider equation (1) with $p = 6$, $\kappa = m = 1$ and $x_0 = 0$. On application of equations (3), (4) and (5) and item (vi) in Subsection 2.2, we have

$$\sum_{n=0}^{\infty} \phi_n(x) = A - \frac{1}{6}Ax^2 + \frac{1}{24}Ax^4 - \frac{83}{6480}Ax^6 + \ldots$$

which is the multivariate Taylor series of the exact solitary wave where $A = \sqrt[6]{112}$. More results are shown in Figure 1.

Case 2. Octic nonlinearity

Consider equation (1) with $p = 7$, $\kappa = m = 1$ and $x_0 = 0$. Similarly, on application of equations (3), (4) and (5) and item (vii) in Subsection 2.2, we get

$$\sum_{n=0}^{\infty} \phi_n(x) = A - \frac{1}{7} Ax^2 + \frac{5}{147} Ax^4 - \frac{158}{15435} Ax^6 + \ldots$$

which is also the multivariate Taylor series of the exact soliton. Where $A = \sqrt[7]{144}$. More results are shown in Figure 2.

Case 3. Nonic nonlinearity

Consider equation (1) with $p = 8$, $\kappa = m = 1$ and $x_0 = 0$. Also, on application of equations (3), (4) and (5) and item (viii) in Subsection 2.2, we obtained

$$\sum_{n=0}^{\infty} \phi_n(x) = A - \frac{1}{8} Ax^2 + \frac{11}{384} Ax^4 - \frac{391}{46080} Ax^6 + \ldots$$

which is the same as the multivariate Taylor series of the exact wave profile. Where $A = \sqrt[8]{180}$. More results are shown in Figure 3.

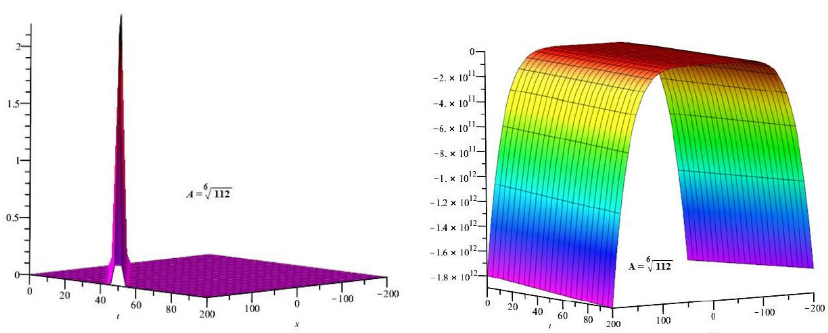

Figure 1. Exact solution versus finite terms of ADM result.

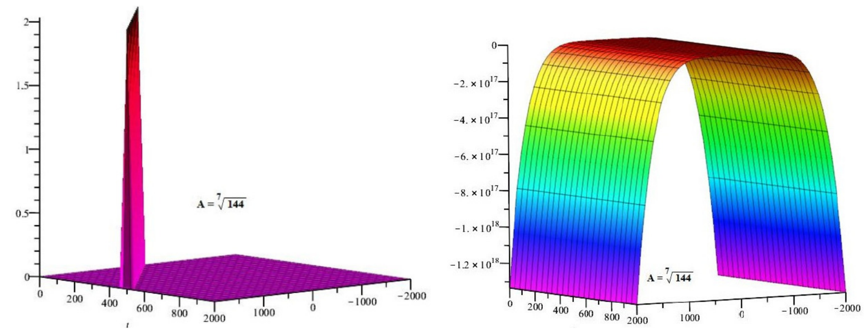

Figure 2. Exact solution versus finite terms of ADM result.

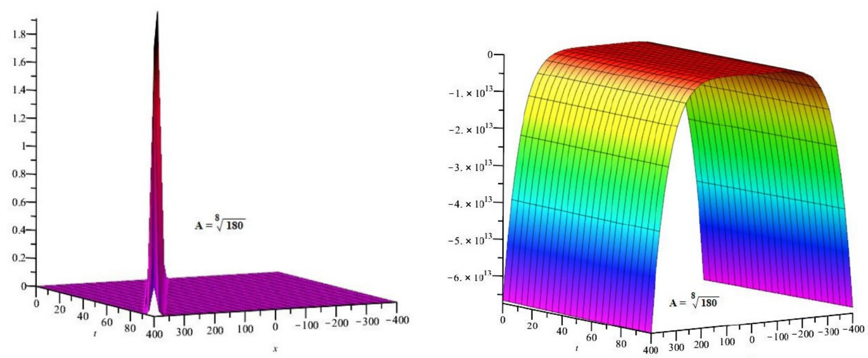

Figure 3. Exact solution versus finite terms of ADM result.

4. Conclusion

ADM has and will continue to be an important powerful tool for obtaining closed and numerical result to wide ranging classes of mathematical problems with a proven record of reliability. However, its successful deployment in nonlinear analysis is heavily dependent on 'the right' APs used. In this paper, we have demonstrated the use of enhanced ADM to attain continuous analytically exact results to the gKdV model in multivariate Taylor series form. This was made possible by the APs of the advection nonlinearity term we gave unambiguously from the least case scenerio, in this problem, to the nonic term. Then, expressed the initial conditions in the single variable Taylor series and applied each term in the integral equation representing each component of the wave profile. And, we gave illustrations whose 3D plots appeared to be similar in the three cases we considered.

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